21-259: Calculus in Three Dimensions Review Sheet

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1. Derivatives and Integrals

1.1 Trigonometric Identities

- Inverses:
  \[ \csc x = \frac{1}{\sin x} \]
  \[ \sec x = \frac{1}{\cos x} \]
  \[ \cot x = \frac{1}{\tan x} \]

- Pythagorean: \( \sin^2 x + \cos^2 x = 1 \)

- Quotient: \( \tan x = \frac{\sin x}{\cos x} \)
• Double angle:
  \[ \sin(2x) = 2\sin x \cos x \]
  \[ \cos(2x) = \cos^2 x - \sin^2 x \]
  
  All the useful identities can be derived easily from these. See Full Table of Trig Identities.

1.2 Derivatives

You should have this Table of Derivatives memorized, except hyperbolics.

- Product rule: \[ \frac{d}{dx} f(x)g(x) = f'(x)g(x) + g'(x)f(x) \]
- Quotient rule: \[ \frac{d}{dx} \frac{f(x)}{g(x)} = \frac{f'(x)g(x) - g'(x)f(x)}{(g(x))^2} \]
- Chain rule: \[ \frac{d}{dx} f(g(x)) = f'(g(x))g'(x) \]
- Implicit differentiation

1.3 Integrals

See Irina’s Integration Practice and Solutions (Andrew login required).

- \( u \)-substitution (problem 1 above)
- Trig identity substitution (problem 3)
- Integration by parts (problems 2, 4)
  
  Can be applied multiple times. Note the “two-sided” technique in 4.

- Integration by partial fractions (problems 5, 6)
- Trig substitution (rare, but good to know)

2 Vectors and the Geometry of Space (Stewart’s Chapter 12)

2.1 Vectors

- Unit vector in the direction of \( \vec{v} \) is \( \vec{u} = \frac{\vec{v}}{|\vec{v}|} \)
- Orthogonal means perpendicular

2.1.1 Dot Product

- \( \vec{a} \cdot \vec{b} = |\vec{a}||\vec{b}| \cos \theta \)
- Scalar projection of \( \vec{b} \) onto \( \vec{a} \): \( \text{comp}_\vec{a}(\vec{b}) = \frac{\vec{a} \cdot \vec{b}}{|\vec{a}|} \)
- Vector projection of \( \vec{b} \) onto \( \vec{a} \): \( \text{proj}_\vec{a}(\vec{b}) = \text{comp}_\vec{a}(\vec{b}) \frac{\vec{a}}{|\vec{a}|} \)

2.1.2 Cross Product

- \[ |\vec{a} \times \vec{b}| = |\vec{a}||\vec{b}| \sin \theta \]
- \( \vec{a} \) and \( \vec{b} \) are parallel iff \( \vec{a} \times \vec{b} = \vec{0} \)
- The area of the parallelogram determined by \( \vec{a} \) and \( \vec{b} \) is \( A = |\vec{a} \times \vec{b}| \)
2.1.3 Triple Product

- The volume of the parallelepiped determined by \( \vec{a} \), \( \vec{b} \), and \( \vec{c} \) is \( V = \vec{a} \cdot (\vec{b} \times \vec{c}) = (\vec{a} \times \vec{b}) \cdot \vec{c} \)

2.2 Lines

- Parallel lines have proportional direction vectors
- Skew lines neither intersect nor are parallel
  
  Check for intersection by setting the parametric equations equal and solving the system

2.2.1 Equations of Lines

- Vector equation: \( \vec{r} = \vec{r}_0 + t\vec{v} \), where \( \vec{r}_0 \) is the vector from the origin to any point on the line, and \( \vec{v} \) is the direction vector of the line
- Parametric equations: \( x = x_0 + at \), \( y = y_0 + bt \), \( z = z_0 + ct \)
  
  \( a, b, c \) are the direction numbers of the line; i.e. \( \vec{v} = \langle a, b, c \rangle \)
- Symmetric equations: \( \frac{x-x_0}{a} = \frac{y-y_0}{b} = \frac{z-z_0}{c} \)
- Line segment from \( \vec{r}_0 \) to \( \vec{r}_1 \): \( \vec{r}(t) = (1-t)\vec{r}_0 + t\vec{r}_1, 0 \leq t \leq 1 \)

2.3 Planes

- A plane is defined by a point \( P(x_0, y_0, z_0) \) and a normal vector \( \vec{n} = \langle a, b, c \rangle \)
- Parallel planes have parallel (proportional) normal vectors

2.3.1 Equations of Planes

- Scalar equation: \( a(x-x_0) + b(y-y_0) + c(z-z_0) = 0 \)
- Linear equation: \( ax + by + cz + d = 0 \)

2.3.2 Distance between a Point and a Plane

This distance is equal to the scalar projection of a vector \( \vec{b} \) from a point \( P_0 \) on the plane to the given point \( P_1(x_1, y_1, z_1) \) onto the plane’s normal vector \( \vec{n} = \langle a, b, c \rangle \). So
  
  \[ D = |\text{comp}_{\vec{n}}\vec{b}| = \frac{|ax_1+by_1+cz_1+d|}{\sqrt{a^2+b^2+c^2}} \]

2.4 Quadric Surfaces

2.4.1 Equations of Quadric Surfaces

Note: Everywhere we have \( x, y, z \) in these formulae we can replace with \( x-h, y-k, z-p \) to shift the center of the surface from the origin to \( (h, k, p) \). Of course, we can also permute \( x, y, z \) to get orientations around different axes.

See Stewart’s 7E page 830 for a chart with pictures.

- Ellipsoid: \( \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1 \)
- Cone: \( \frac{z^2}{x^2} = \frac{a^2}{b^2} \)
• Elliptic Paraboloid: \( z = \frac{x^2}{a^2} + \frac{y^2}{b^2} \)

• Hyperboloid of One Sheet: \( \frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1 \)

• Hyperbolic Paraboloid: \( \frac{z^2}{a^2} = \frac{x^2}{a^2} - \frac{y^2}{b^2} \)

• Hyperboloid of Two Sheets: \( -\frac{x^2}{a^2} - \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1 \)

3 Vector Functions (Stewart’s Chapter 13)

3.1 Vector Functions and Space Curves

• A vector function \( \mathbf{r}(t) = \langle f(t), g(t), h(t) \rangle \) is continuous at \( a \) if

\[
\lim_{t \to a} \mathbf{r}(t) = \mathbf{r}(a)
\]

• The set of points \( (f(t), g(t), h(t)) \) on an interval of the parameter \( t \) is a space curve.

\( x = f(t), y = g(t), z = h(t) \) are the parametric equations of this space curve.

3.2 Derivatives of Vector Functions

Let \( C \) be the curve defined by \( \mathbf{r} \). Then

• \( \mathbf{r}'(t) \) is the tangent vector to \( C \)

• The tangent line to \( C \) at a point \( P \) is the line through \( P \) and parallel to \( \mathbf{r}'(t) \)

• The unit tangent vector is \( \mathbf{T}(t) = \frac{\mathbf{r}'(t)}{||\mathbf{r}'(t)||} \)

• The unit normal vector is \( \mathbf{N}(t) = \frac{\mathbf{T}'(t)}{||\mathbf{T}'(t)||} \)

• The binormal vector is \( \mathbf{B}(t) = \mathbf{T}(t) \times \mathbf{N}(t) \)

3.3 Arc Length

• The length of the space curve defined by \( \mathbf{r}(t) = \langle x(t), y(t), z(t) \rangle \) on the interval \( a \leq t \leq b \) is

\[
L = \int_a^b ||\mathbf{r}'(t)|| \, dt = \int_a^b \sqrt{\left( \frac{dx}{dt} \right)^2 + \left( \frac{dy}{dt} \right)^2 + \left( \frac{dz}{dt} \right)^2} \, dt
\]

3.3.1 The Arc Length Function

• The arc length function of \( \mathbf{r}(t) \) on the interval \( a \leq t \leq b \) is

\[
s(t) = \int_a^t ||\mathbf{r}'(u)|| \, du = \int_a^t \sqrt{\left( \frac{dx}{du} \right)^2 + \left( \frac{dy}{du} \right)^2 + \left( \frac{dz}{du} \right)^2} \, du
\]

• Differentiating gives \( \frac{ds}{dt} = ||\mathbf{r}'(t)|| \)

3.3.2 Reparametrizing w/ Arc Length

• Use the arc length function to solve for \( t \) in terms of \( s \). Then substitute \( t(s) \) in for \( t \) in \( \mathbf{r}(t) \).
3.4 Curvature

- A parametrization of a curve \( \vec{r}(t) \) on an interval \( I \) is smooth if \( \vec{r}' \) is continuous and \( \vec{r}'(t) \neq 0 \) on \( I \).
- The curvature of a curve is

\[
\kappa(t) = \frac{d\vec{T}}{ds} = \frac{\lVert \vec{T}'(t) \rVert}{\lVert \vec{r}'(t) \rVert} = \frac{|\vec{r}'(t) \times \vec{r}''(t)|}{|\vec{r}'(t)|^3}
\]

- For a plane curve \( y = f(x) \), \( \kappa(x) = \frac{|f''(x)|}{[1+(f'(x))^2]^{3/2}} \)

4 Multivariable Functions (Stewart’s Chapter 14)

4.1 Level Curves and Surfaces

- The level curves of a function \( f(x, y) \) are the curves \( f(x, y) = k \) for all constant \( k \) in the image of \( f \) (think contour map)
- The level surfaces of a function \( f(x, y, z) \) are the curves \( f(x, y, z) = k \) for all constant \( k \) in the image of \( f \)

4.2 Limits and Continuity

- Technical definition: Let \( D \) be the domain of a function \( f \) which includes points arbitrarily close to \( (a, b) \). Then

\[
\lim_{(x,y)\to(a,b)} f(x, y) = L
\]

iff

\[
\forall (\epsilon > 0). \exists (\delta > 0). ((x, y) \in D \land 0 < \sqrt{(x-a)^2 + (y-b)^2} < \delta) \implies |f(x, y) - L| < \epsilon
\]
- A function is continuous at \( (a, b) \) if

\[
\lim_{(x,y)\to(a,b)} f(x, y) = f(a, b)
\]

4.2.1 Techniques for Finding and Proving Limits

See Irina’s Limits Handout for examples of using these techniques (requires Andrew login).

- Direct substitution (when \( f \) is a rational function)
- Fancy algebra (factoring, cancelling, etc.)
- L’Hôpital’s rule (when direct substitution yields \( \frac{0}{0} \) or \( \pm\infty/\infty \))
- Squeeze theorem (bound the function on both sides, and show that the limits of these bounds are equal)
- Apply the definition

4.2.2 The Limit Does Not Exist!

Show that approaching from two different paths gives different limits.
4.3 Partial Derivatives
- $f$ is differentiable at $(a,b)$ if $f_x$ and $f_y$ exist and are continuous near $(a,b)$
- Clairaut’s theorem: if $f$ is defined on a disk $D$ containing $(a,b)$ and $f_{xy}$ and $f_{yx}$ are continuous on $D$, $f_{xy}(a,b) = f_{yx}(a,b)$

4.4 Tangent Planes
4.4.1 Finding an Equation of a Tangent Plane
- If $f$ has continuous partial derivatives, then an equation of the tangent plane to the surface $z = f(x,y)$ at $P(x_0, y_0, z_0)$ is $z - z_0 = f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)$

4.4.2 Linear Approximations
- The linearization of $f$ at $(a,b)$ is $L(x,y) = f(a,b) + f_x(a,b)(x - a) + f_y(a,b)(y - b)$
- A linear approximation of $f$ near $(a,b)$ is $f(x,y) \approx L(x,y)$
- The differential $dz$ is $dz = f_x(x,y)dx + f_y(x,y)dy$ (think error in $z$ given error in $x$ and $y$)

4.5 Chain Rule
- If $z = f(x(t), y(t))$, then $\frac{dz}{dt} = \frac{∂z}{∂x} \frac{dx}{dt} + \frac{∂z}{∂y} \frac{dy}{dt}$
- Implicit Function Theorem: under the logical conditions, when we have an equation $F(x,y) = 0$, $\frac{dy}{dx} = -\frac{F_x}{F_y}$
- Or if $F(x,y,z) = 0$, then $\frac{∂z}{∂x} = -\frac{F_x}{F_z}$, $\frac{∂z}{∂y} = -\frac{F_y}{F_z}$

4.6 Directional Derivatives
- If $f(x,y)$ is differentiable, then the directional derivative of $f$ in the direction of a unit vector $\vec{u} = \langle a, b \rangle$ is $D_u f(x,y) = f_x(x,y)a + f_y(x,y)b = \nabla f \cdot \vec{u}$

4.7 The Gradient Vector
- The gradient of $f(x,y,z)$ is $\nabla f = \langle f_x, f_y, f_z \rangle$
- The direction of $\nabla f$ is the direction of fastest change in $f$

4.8 Local and Absolute Extrema
- A point $(a,b)$ is a critical point of $f$ if $f_x(a,b) = 0$ and $f_y(a,b) = 0$
- A local extremum must be a critical point, but not all critical points are extrema
4.8.1 The Second Derivatives Test
This test determines whether a critical point is a local extremum. If \( f \) has continuous second partial derivatives and \((a, b)\) is a critical point of \( f \), let

\[
D = f_{xx}f_{yy} - f_{xy}^2
\]

- If \( D(a, b) > 0 \) and \( f_{xx}(a, b) > 0 \), then \((a, b)\) is a local minimum point
- If \( D(a, b) > 0 \) and \( f_{xx}(a, b) < 0 \), then \((a, b)\) is a local maximum point
- If \( D(a, b) < 0 \), then \((a, b)\) is a saddle point
- When \( D(a, b) = 0 \), the test is inconclusive

4.8.2 Absolute Extrema
To find the absolute extrema of \( f \) on a closed set \( D \), calculate the values of \( f \) at its critical points and along the boundaries of \( D \) (use the derivative and number line technique for finding extrema along boundary curves).

- The absolute maximum is the largest of these values
- The absolute minimum is the smallest of these values

4.9 Lagrange Multipliers
The Lagrange method maximizes a function \( f(x, y, z) \) subject to one or two constraints \( g(x, y, z) = k \) and \( h(x, y, z) = c \).

1. Set \( \nabla f = \lambda \nabla g + \mu \nabla h \), yielding the system of equations:
   \[
   \begin{align*}
   f_x &= \lambda g_x + \mu h_x \\
   f_y &= \lambda g_y + \mu h_y \\
   f_z &= \lambda g_z + \mu h_z \\
   g(x, y, z) &= k \\
   h(x, y, z) &= c
   \end{align*}
   \]
2. Find the values of \( f \) at all points \((x, y, z)\) which satisfy this system
3. The maximum of these is the absolute maximum value and the minimum of these is the absolute minimum value of \( f \) subject to the constraint(s)

5 Multiple Integrals (Stewart’s Chapter 15)
Note: Remember that axes can be permuted, and the best order of integrals to take may change.

5.1 Double Integrals
Procedure:
1. Choose the most convenient coordinate system for the bounds
   - Rectangular (normal)
• Polar:
  \[ x = r \cos \theta \]
  \[ y = r \sin \theta \]

2. Evaluate the iterated integral \( \iint_D f(x, y) \, dA \)

  • Rectangular: \( \int_a^b \int_{g_1(x)}^{g_2(x)} f(x, y) \, dy \, dx \)
  • Polar: \( \int_\alpha^\beta \int_{h_1(\theta)}^{h_2(\theta)} f(r \cos \theta, r \sin \theta) \, r \, dr \, d\theta \)

5.1.1 Surface Area

The area of the surface \( z = f(x, y) \), \((x, y) \in D\), where \( f_x \) and \( f_y \) are continuous, is

\[
A = \iint_D \sqrt{1 + \left( \frac{\partial z}{\partial x} \right)^2 + \left( \frac{\partial z}{\partial y} \right)^2} \, dA
\]

5.2 Triple Integrals

Procedure:

1. Choose the most convenient coordinate system for the bounds
   - Rectangular
   - Cylindrical: as polar above, plus \( z = z \)
   - Spherical:
     \[ z = \rho \cos \phi \]
     \[ r = \rho \sin \phi \]
     \[ \rho^2 = x^2 + y^2 + z^2 \]

2. Evaluate the iterated integral \( \iiint_E f(x, y, z) \, dV \)
   - Rectangular: \( \int_a^b \int_{g_1(x)}^{g_2(x)} \int_{u_1(y)}^{u_2(y)} f(x, y, z) \, dz \, dy \, dx \)
   - Cylindrical: \( \int_\alpha^\beta \int_{h_1(\theta)}^{h_2(\theta)} \int_{u_1(r \cos \theta, r \sin \theta)}^{u_2(r \cos \theta, r \sin \theta)} f(r \cos \theta, r \sin \theta, z) \, r \, dz \, dr \, d\theta \)
   - Spherical: \( \int_c^d \int_\alpha^\beta \int_a^b f(x, y, z) \rho^2 \sin \phi \, d\rho \, d\theta \, d\phi \)

6 Vector Calculus (Stewart’s Chapter 16)

See Stewart’s 7E page 1135 for a one page chart (with pictures) summary of how the main theorems of this chapter relate.

6.1 Vector Fields

• A vector field is a function \( \vec{F} \) that assigns to each point \((x, y, z)\) a vector \( \vec{F}(x, y, z) \)

• We can write a vector field in terms of its component scalar functions: \( \vec{F}(x, y, z) = P(x, y, z)i + Q(x, y, z)j + R(x, y, z)k \)
6.2 Conservative Fields

- A vector field $\vec{F}$ is **conservative** if it is the gradient of some scalar function $f$; i.e. $\vec{F} = \nabla f$
  
  $f$ is called the **potential function** for $\vec{F}$

6.2.1 The Fundamental Theorem for Line Integrals

Let $C$ be a smooth curve defined by $\vec{r}(t)$ on the interval $a \leq t \leq b$. If $f$ is a differentiable function with a continuous gradient vector on $C$, then

$$\int_C \nabla f \cdot d\vec{r} = f(\vec{r}(b)) - f(\vec{r}(a))$$

So the line integral of a conservative vector field over a curve $C$ is simply the net change in the potential function between the endpoints of $C$; i.e. the integral is **independent of path**.

6.2.2 Determining Conservatism

- Suppose $\vec{F} = P\vec{i} + Q\vec{j}$ is a vector field on an open, simply-connected region $D$, and $P$ and $Q$ have continuous first-order derivatives. If
  $$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$$
  throughout $D$, then $\vec{F}$ is conservative

- If $\vec{F}$ is conservative, then $\text{curl} \; \vec{F} = 0$. Further, if $\vec{F}$ is defined on all of $\mathbb{R}^3$ and its component functions have continuous partial derivatives, then the converse is true

6.2.3 Finding the Potential Function

Because $\vec{F} = \nabla f$, we have $f_x = P$, $f_y = Q$, and $f_z = R$. We use partial integration and differentiation, comparing with $P$, $Q$, and $R$, to solve for the function $f(x, y, z)$.

6.3 Line Integrals

6.3.1 Evaluating Line Integrals over Plane Curves

- For continuous $f$, the line integral (wrt arc length) over the curve $C$ on the interval $a \leq t \leq b$ is
  $$\int_C f(x, y) \, ds = \int_a^b f(x(t), y(t)) \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} \, dt$$

  - With respect to $x$, $\int_C f(x, y) \, dx = \int_a^b f(x(t), y(t)) x'(t) \, dt$
  - With respect to $y$, $\int_C f(x, y) \, dy = \int_a^b f(x(t), y(t)) y'(t) \, dt$

6.3.2 Evaluating Line Integrals over Space Curves

- For continuous $f$, the line integral (wrt arc length) over the curve $C$ on the interval $a \leq t \leq b$ is
  $$\int_C f(x, y, z) \, ds = \int_a^b f(x(t), y(t), z(t)) \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2} \, dt$$

  - With respect to $x$, $\int_C f(x, y, z) \, dx = \int_a^b f(x(t), y(t), z(t)) x'(t) \, dt$
  - With respect to $y$, $\int_C f(x, y, z) \, dy = \int_a^b f(x(t), y(t), z(t)) y'(t) \, dt$
  - With respect to $z$, $\int_C f(x, y, z) \, dz = \int_a^b f(x(t), y(t), z(t)) z'(t) \, dt$
6.3.3 Line Integrals of Vector Fields

If $\vec{F} = P\vec{i} + Q\vec{j} + R\vec{k}$ is a continuous vector field on the curve $C$ defined by the vector $\vec{r}(t)$ on the interval $a \leq t \leq b$, the line integral of $\vec{F}$ along $C$ is

$$\int_C \vec{F} \cdot d\vec{r} = \int_C P\, dx + Q\, dy + R\, dz$$

6.4 Green’s Theorem

- The positive orientation of a simple, closed curve is a single counterclockwise traversal of the curve.
- Let $C$ be a positively-oriented, piecewise-smooth, simple closed curve in the plane and let $D$ be the region bounded by $C$. If $P$ and $Q$ have continuous partial derivatives on $D$, then

$$\int_C P\, dx + Q\, dy = \iint_D \left( \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA$$

6.4.1 Vector Forms of Green’s Theorem

- $\int_C \vec{F} \cdot d\vec{r} = \iint_D (\text{curl } \vec{F}) \cdot \hat{k} \, dA$
- $\int_C \vec{F} \cdot \hat{n} \, ds = \iint_D \text{div } \vec{F} \, dA$ where $\hat{n}$ is the outward unit normal vector to $C$.

6.5 Curl

Recall the definition $\nabla = \frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}$. Then the curl of a vector field $\vec{F}$ is

$$\text{curl } \vec{F} = \nabla \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{vmatrix}$$

6.6 Divergence

The divergence of a vector field $\vec{F}$ is

$$\text{div } \vec{F} = \nabla \cdot \vec{F} = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}$$

assuming these partial derivatives exist. Note that $\text{div } \vec{F}$ is a scalar field.

- If $P$, $Q$, and $R$ have continuous second-order partial derivatives, then $\text{div } \text{curl } \vec{F} = 0$.
  It follows that if $\text{div } \vec{F} \neq 0$, $\vec{F}$ cannot be written as the curl of another vector field.

6.7 Parametric Surfaces

- The set of points $(x, y, z)$ traced out by $\vec{r}(u, v) = x(u, v)\vec{i} + y(u, v)\vec{j} + z(u, v)\vec{k}$ as $(u, v)$ varies throughout a region $D$ is a parametric surface.
- Holding $u$ or $v$ constant gives the grid curves of $\vec{r}(u, v)$.
6.7.1 Finding Equations of Parametric Surfaces

1. Choose a coordinate system where one of the variables is constant or can be written as a function of the other two
   - Rectangular: e.g. when given \( z = f(x, y) \)
   - Polar: e.g. when \( z \) can be written \( z = f(r, \theta) \)
   - Spherical: e.g. when constant radius \( \rho = a \)

2. Write the vector function \( \vec{r}(u, v) \) in terms of the two variables

6.7.2 Surfaces of Revolution

Example: given a function of the form \( y = f(x) \), find parametric equations for the surface generated by rotating \( y = f(x) \) about the \( x \)-axis. Take \( \theta \) as the second parameter, and write \( x = x, y = f(x) \cos \theta, z = f(x) \sin \theta \).

6.7.3 Tangent Planes to Parametric Surfaces

To find the tangent plane to a surface \( \vec{r}(u, v) \) at \( P(x, y, z) \),

1. Find the tangent vectors \( \vec{r}_u \) and \( \vec{r}_v \)
2. Compute the normal vector \( \vec{r}_u \times \vec{r}_v \)
3. Find the point \( (u_0, v_0) \) which corresponds to \( (x, y, z) \)
4. Plug and chug into the scalar plane equation

6.7.4 Surface Area

If \( S \) is a smooth parametric surface \( (\vec{r}_u \times \vec{r}_v \neq \vec{0}) \) and is covered only once as \( (u, v) \) ranges throughout \( D \), then the surface area of \( S \) is

\[
A = \iint_D |\vec{r}_u \times \vec{r}_v| dA
\]

6.8 Surface Integrals

The surface integral of \( f \) over a parametric surface \( S \) is

\[
\iint_D f(\vec{r}(u, v))|\vec{r}_u \times \vec{r}_v| dA
\]

where \( D \) is the region over which \( (u, v) \) ranges. Note: we DON’T need to add the \( r \) or \( \rho^2 \sin \phi \) when using polar or spherical coordinates in this case!

For graphs, i.e. functions of the form \( f(x, y, g(x, y)) \) modulo axis permutation, this becomes

\[
\iint_D f(x, y, g(x, y)) \sqrt{\left( \frac{\partial z}{\partial x} \right)^2 + \left( \frac{\partial z}{\partial y} \right)^2 + 1} \ dA
\]

6.8.1 Surface Integrals of Vector Fields

If \( \vec{F} \) is a continuous vector field defined on an oriented surface \( S \) with unit normal vector \( \vec{n} \), then the surface integral of \( \vec{F} \) over \( S \) (the flux of \( \vec{F} \) across \( S \)) is

\[
\iint_S \vec{F} \cdot d\vec{S} = \iint_S \vec{F} \cdot \vec{n} \ dS = \iint_D \vec{F} \cdot (\vec{r}_u \times \vec{r}_v) \ dA
\]
6.9  Stokes’ Theorem

Let $S$ be an oriented, piecewise-smooth surface bounded by a simple, closed, piecewise-smooth curve $C$ with positive orientation. Let $\vec{F}$ be a vector whose components have continuous partial derivatives on $S$. Then

$$\int_C \vec{F} \cdot d\vec{r} = \iint_S \text{curl} \vec{F} \cdot d\vec{S}$$

6.10  Divergence Theorem

Let $E$ be a simple solid region and let $S$ be the boundary surface of $E$ with outward orientation. Let $\vec{F}$ be a vector field whose components have continuous partial derivatives on $E$. Then

$$\iint_S \vec{F} \cdot d\vec{S} = \iiint_E \text{div} \vec{F} \, dV$$

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